

Octonionic Version of Dirac Equations

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Abstract

It is shown that a simple continuity condition in the algebra of split octonions suffices to formulate a system of differential equations that are equivalent to the standard Dirac equations. In our approach the particle mass and electro-magnetic potentials are part of an octonionic gradient function together with the space-time derivatives. As distinct from previous attempts to translate the Dirac equations into different number systems here the wave functions are real split octonions and not bi-spinors. To formulate positively defined probability amplitudes four different split octonions (transforming into each other by discrete transformations) are necessary, rather than two complex wave functions which correspond to particles and antiparticles in usual Dirac theory.

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1 Introduction

One of the breakthroughs in the development of field theory was the discovery of the Dirac equation in 1928. The question we wish to address in this article is whether one can formulate the Dirac equation without availing oneself of complex bi-spinors and matrix algebra, and to what extent such a formulation can be brought into a form equivalent to the standard theory. The successful application of quaternions [1] and Geometric Algebras [2] in formulating a Dirac equation without matrices, initiated the use of octonions as underlying numerical fields [3, 4]. Real octonions also contain eight parameters, just as Dirac bi-spinors, or complex quaternions.

Octonions form the widest normed algebra after the algebras of real numbers, complex numbers, and quaternions [5]. Since their discovery in 1844-1845 by Graves

and Cayley there have been various attempts to find appropriate uses for octonions in physics (see reviews [6]). One can point to the possible impact of octonions on: Color symmetry [7]; GUTs [8]; Representation of Clifford algebras [9]; Quantum mechanics [10]; Space-time symmetries [11]; Field theory [12, 13]; Formulations of wave equations [3, 4, 14]; Quantum Hall effect [15]; Strings and M -theory [16]; *etc.*

The structure of the matrices in the Dirac equation is linked to relativistic covariance. However, space-time geometry does not only have to be formulated using standard Lorentz four-vectors, matrices, or quaternionic notation, but can also be formulated using an octonionic parametrization.

In our previous papers [17] it was introduced the concept of Octonionic Geometry based on the algebra of split octonions. This approach is related with so-called Geometric Algebras [18] in the sense that we also emphasized the geometric significance of vectors (which are more effective than spinors and tensors in conveying geometry) and avoided matrices and tensors.

In the present paper we shall show that the algebra of split octonions, we used in [17] to describe the geometry, suffice to formulate a system of differential equations equivalent to the standard Dirac equations.

2 Octonionic Geometry

The geometry of space-time in the language of algebras and symmetries can be described. Any observable quantity, which our brain could extract from a single measurement is a real number. Introduction of the distance (norm) always means some comparison of two physical objects using one of them as an etalon. In the algebraic language these features mean that to perceive the real world our brain uses normed algebras with the unit element over the field of real numbers. In physical applications of normed algebras mainly the elements with the negative square (which are similar to ordinary complex unit) are used. In this case norm of the algebra is positively defined. Introduction of vector-like elements with positive square and negative norm leads to so-called split algebras. Because of pseudo-Euclidean character of there norms split-algebras are useful to study dynamics.

In the paper [17] it was assumed that to describe the geometry of real world most convenient is the algebra of split-octonions. With a real physical signal we associate an 8-dimensional number, the element of split octonions,

$$s = ct + x^n J_n + \hbar \lambda^n j_n + c \hbar \omega I . \quad (n = 1, 2, 3) \quad (1)$$

Some characteristics of the physical world (such as dimension, causality, maximal velocities, quantum behavior, *etc.*) can be naturally connected with the structure of the algebra. For example, our imagination about 3-dimensional character of the space can be the result of existing of the three vector like elements J_n in (1).

We interpret the basis elements of split octonions as multi-vectors, similar to Geometric Algebras [18]. In (1) the scalar unit is denoted as 1, the three vector-like objects as J_n , the three pseudo-vectors as j_n and the pseudo-scalar as I . The eight scalar parameters that multiply the basis units in (1) we treat as the time t , the

special coordinates x^n , the wavelength λ^n and the frequency ω . The quantity (1) also contains two fundamental constants of physics - the velocity of light c and Planck's constant \hbar . The appearance of these constants is connected with the existence of two classes of zero divisors in the algebra of split octonions [17].

The algebra of the basis elements of split octonions can be written in the form:

$$\begin{aligned} J_n^2 &= -j_n^2 = I^2 = 1 , \\ J_n j_m &= -j_m J_n = -\epsilon_{nmk} J^k , \\ J_n J_m &= -J_m J_n = j_n j_m = -j_m j_n = \epsilon_{nmk} j^k , \\ J_n I &= -I J_n = j_n , \\ j_n I &= -I j_n = J_n , \end{aligned} \tag{2}$$

where ϵ_{nmk} is the fully antisymmetric tensor and $n, m, k = 1, 2, 3$. From these formulae it can be seen that to generate a complete 8-dimensional basis of split octonions the multiplication and distribution laws of only three vector-like elements, J_n , are enough. The other two basis units j_n and I can be expressed as binary and triple products

$$j_n = \frac{1}{2} \epsilon_{nmk} J^m J^k , \quad I = J_n j_n \tag{3}$$

(there is no summing of indices in the second formula).

The essential property of octonions, non-associativity, is the direct result of the second formula of (3). Since the 3-vector I has three equivalent representations we find, for example,

$$J_1(J_2 J_3) - (J_1 J_2) J_3 = J_1 j_1 - j_3 J_3 = 2I \neq 0 . \tag{4}$$

We adopt the non-associativity of the triple products of J_n , and at the same time we need to have definite results for the multiplication of all seven octonionic basis units (2). For this purpose the property of alternativity of octonions can be used. This weak form of associativity implies the Moufang identities for the products of any four element when two of them coincide [5]

$$(ax)(ya) = a(xy)a , \quad a(x(ay)) = (axa)y , \quad y(a(xa)) = y(axa) . \tag{5}$$

In physical applications we interpret the non-associativity of octonions, which results in the non-equivalence of left and right products for expressions containing more than two basis units J_n , as corresponding to causality [17]. For the direction from the past to the future we want to use one definite order of multiplication, for example the left product. Then non-associativity leads to the appearance of time asymmetries in our model.

The standard conjugation of fundamental vector-like basis units

$$J_n^* = -J_n , \tag{6}$$

can be imagined as reflections. Analogous to Clifford algebras [2] we introduce three different kind of conjugations (involutions) of products of several J_n and thus

define conjugations of other octonionic basis units j_i and I . The standard octonionic anti-automorphism

$$(J_i J_k J_n \dots)^* = \dots J_n^* J_k^* J_i^* , \quad (7)$$

reverses the order of elements in any given expression.

We can define also the following automorphism (conjugation without reversion)

$$(J_i J_k J_n \dots)^\dagger = J_i^* J_k^* J_n^* \dots , \quad (8)$$

and combined involution

$$\overline{(J_i J_k J_n \dots)} = (J_i J_k J_n \dots)^{\dagger*} . \quad (9)$$

Similar involutions in Clifford algebras are called respectively conjugation, grade involution and reversion [2].

Since we interpret the conjugation of vector-like elements J_n as reflection and the directivity feature of their products as corresponding to the time arrow, these three involutions (7), (8) and (9) can be considered analogously to the discrete symmetries T , P and C .

The involutions (7), (8) and (9) do not affect the unit elements of split octonions, while other basis elements change according to the laws

$$\begin{aligned} J_i^* &= -J_i , & j_i^* &= -j_i , & I^* &= -I , \\ J_i^\dagger &= -J_i , & j_i^\dagger &= j_i , & I^\dagger &= -I , \\ \overline{J_i} &= J_i , & \overline{j_i} &= -j_i , & \overline{I} &= I . \end{aligned} \quad (10)$$

The principal conjugation of octonions (7) is usually used to define their norm. For example, conjugation of (1) gives

$$s^* = ct - x_n J^n - \hbar \lambda_n j^n - c \hbar \omega I . \quad (11)$$

Then the norm of (1)

$$s^2 = ss^* = s^*s = c^2 t^2 - x_n x^n + \hbar^2 \lambda_n \lambda^n - c^2 \hbar^2 \omega^2 , \quad (12)$$

has a $(4 + 4)$ signature and in the limit $\hbar \rightarrow 0$ gives the classical formula for the Minkowski interval.

It is possible to give a representation of the basis units of octonions $(1, J_n, j_n, I)$ through 2×2 Zorn matrices [5], whose diagonal elements are scalars and whose off-diagonal elements are 3-dimensional vectors

$$\begin{aligned} 1 &\Longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & J_n &\Longleftrightarrow \begin{pmatrix} 0 & \sigma_n \\ \sigma_n & 0 \end{pmatrix}, \\ I &\Longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & j_n &\Longleftrightarrow \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix}. \end{aligned} \quad (13)$$

Here the elements σ_n ($n = 1, 2, 3$), with the property $\sigma_n^2 = 1$ can be considered as ordinary Pauli matrices. Of course one can use the real representation also, with the complex Pauli matrix σ_2 replaced by the unit matrix.

Using (13) the split octonion (1) can be written as

$$s = ct + x_n J^n + \hbar \lambda_n j^n + c \hbar \omega I = \begin{pmatrix} c(t + \hbar \omega) & (x_n - \hbar \lambda_n) \sigma^n \\ (x_n + \hbar \lambda_n) \sigma^n & c(t - \hbar \omega) \end{pmatrix}. \quad (14)$$

The conjugate of the matrix (14) is defined as

$$s^* = ct - x_n J^n - \hbar \lambda_n j^n - c \hbar \omega I = \begin{pmatrix} c(t - \hbar \omega) & -(x_n - \hbar \lambda_n) \sigma^n \\ -(x_n + \hbar \lambda_n) \sigma^n & c(t + \hbar \omega) \end{pmatrix}. \quad (15)$$

Then the norm (12) is given by the product of these matrices

$$s^* s = (c^2 t^2 - x_n x^n + \hbar^2 \lambda_n \lambda^n - c^2 \hbar^2 \omega^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (16)$$

The two other forms of involution, (8) and (9), for the octonion (1) have the following matrix representations

$$\begin{aligned} s^\dagger &= ct - x_n J^n + \hbar \lambda_n j^n - c \hbar \omega I = \begin{pmatrix} c(t - \hbar \omega) & -(x_n + \hbar \lambda_n) \sigma^n \\ -(x_n - \hbar \lambda_n) \sigma^n & c(t + \hbar \omega) \end{pmatrix}, \\ \bar{s} &= ct + x_n J^n - \hbar \lambda_n j^n + c \hbar \omega I = \begin{pmatrix} c(t + \hbar \omega) & (x_n + \hbar \lambda_n) \sigma^n \\ (x_n - \hbar \lambda_n) \sigma^n & c(t - \hbar \omega) \end{pmatrix}. \end{aligned} \quad (17)$$

Since octonions are not associative, they cannot be represented by matrices with the usual multiplication laws. The product of any matrices written above, have the special multiplication rules [19]

$$\begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} * \begin{pmatrix} \alpha' & \mathbf{a}' \\ \mathbf{b}' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha \alpha' + (\mathbf{a} \mathbf{b}') & \alpha \mathbf{a}' + \beta' \mathbf{a} - [\mathbf{b} \mathbf{b}'] \\ \alpha' \mathbf{b} + \beta \mathbf{b}' + [\mathbf{a} \mathbf{a}'] & \beta \beta' + (\mathbf{b} \mathbf{a}') \end{pmatrix}, \quad (18)$$

where $(\mathbf{a} \mathbf{b})$ and $[\mathbf{a} \mathbf{b}]$ denote the usual scalar and vector products of the 3-dimensional vectors \mathbf{a} and \mathbf{b} . Probably the easiest way to think of this multiplication is to consider the usual matrix product with an added anti-diagonal matrix

$$\begin{pmatrix} 0 & -\epsilon^{nmk} b_m b'_k \\ \epsilon^{nmk} a_m a'_k & 0 \end{pmatrix}. \quad (19)$$

The algebra of octonionic basis units (2) is easily reproduced in this Zorn matrix notation.

Using the algebra of the basis elements (2) the octonion (1) also can be written in the equivalent form

$$s = c(t + \hbar \omega I) + J^n (x_n + \hbar \lambda_n I). \quad (20)$$

From this formula we see that pseudo-scalar I introduces the 'quantum' term corresponding to some kind of uncertainty of the space-time coordinates.

3 Octonionic Dirac Equation

For convenience we take the formulation of ordinary Dirac equation in the notation used in [21]

$$\left(i\hbar\gamma^0\partial_0 + i\hbar\gamma^n\partial_n - \frac{e}{c}\gamma^0A_0 + \frac{e}{c}\gamma^nA_n - mc \right) \Psi = 0 , \quad (21)$$

which is distinct from the standard definition [22] has the opposite sign for γ^0 and scalar potential A_0 . In (21) gamma matrices have the representation

$$\gamma_0 = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma_n = \begin{pmatrix} 0 & \sigma_n \\ \sigma_n & 0 \end{pmatrix} , \quad \gamma_5 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad (22)$$

where σ_n ($n = 1, 2, 3$) are the usual Pauli matrices.

Without any bias about the nature of the quantities involved in the wave-function, including any hidden meaning to the imaginary unit i , the standard 4-dimensional spinor can be written in the form

$$\Psi = \begin{pmatrix} y_0 + iL_3 \\ -L_2 + iL_1 \\ y_3 + iL_0 \\ y_1 + iy_2 \end{pmatrix} . \quad (23)$$

It is characterized by the eight real parameters y^ν and L^ν ($\nu = 0, 1, 2, 3$).

The Dirac equation (21) for the wave-function (23) can be decomposed into an equivalent set of eight real differential equations

$$\begin{aligned} -\frac{e}{c}A^\nu y_\nu + \hbar(F_{03} + f_{21}) &= mcy_0 , \\ \frac{e}{c}A^\nu L_\nu + \hbar(F_{12} + f_{03}) &= mcL_0 , \\ \frac{e}{c}([yA]_{10} + [LA]_{32}) + \hbar(F_{13} + f_{20}) &= mcy_1 , \\ \frac{e}{c}([yA]_{20} + [LA]_{13}) + \hbar(F_{23} + f_{01}) &= mcy_2 , \\ \frac{e}{c}([yA]_{30} + [LA]_{21}) - \hbar\partial^\nu L_\nu &= mcy_3 , \\ \frac{e}{c}([yA]_{32} + [LA]_{01}) + \hbar(F_{02} + f_{13}) &= mcL_1 , \\ \frac{e}{c}([yA]_{13} + [LA]_{02}) + \hbar(F_{10} + f_{23}) &= mcL_2 , \\ \frac{e}{c}([yA]_{12} + [LA]_{03}) - \hbar\partial^\nu y_\nu &= mcL_3 , \end{aligned} \quad (24)$$

where the summing is done by the Minkowski metric $\eta^{\nu\mu}$ ($\nu, \mu = 0, 1, 2, 3$) with the signature $(+ - - -)$ and we have introduced the notations

$$\begin{aligned} f_{\nu\mu} &= \partial_\nu y_\mu - \partial_\mu y_\nu , & [LA]_{\nu\mu} &= L_\nu A_\mu - L_\mu A_\nu , \\ F_{\nu\mu} &= \partial_\nu L_\mu - \partial_\mu L_\nu , & [yA]_{\nu\mu} &= y_\nu A_\mu - y_\mu A_\nu . \end{aligned} \quad (25)$$

Now we want to show that the system (24) can be written as the product of two split octonions with real components.

The particle wave function we denote by the split octonion

$$\psi = -y_0 + y_n J^n + L_n j^n + L_0 I, \quad (n = 1, 2, 3) \quad (26)$$

where we use the same notation for the eight real numbers y^ν and L^ν as for the components of the Dirac wave-function (23). The set of the values of y^ν and L^ν can be understood as the traditional functions depending on the frame, since after any measurement one finds new values for these parameters that depend on the observer's frame.

To the measurement process we want to associate a split octonion having the dimension of momentum

$$\nabla = \hbar \left[c \frac{\partial}{\partial t} + J^n \frac{\partial}{\partial x^n} \right] + \left[- \left(mc + \frac{e}{c} A_0 \right) + \frac{e}{c} A_n J^n \right] I. \quad (27)$$

In this gradient function we use standard notations for the coordinates, mass and components of vector-potential. In the limit $m, A_\nu \rightarrow 0$ the norm of ∇ is the ordinary d'Alembertian.

Now let us write the orientated continuity equation by multiplication of the octonionic wave-function (26) by (27) from the left

$$\nabla_L \psi = \hbar \left[c \frac{\partial}{\partial t} + J^i \frac{\partial}{\partial x^i} \right] \psi + \left[- \left(mc + \frac{e}{c} A_0 \right) + \frac{e}{c} A_n J^n \right] (I\psi) = 0. \quad (28)$$

Because of non-associativity it is crucial in the second term to multiply I first with ψ and then with the remaining terms in the brackets. The orientated product (28) is similar to the barred operators considered in [4].

Using the matrix representation of octonionic basis units (13) the quantities entering the equation (28) can be written in the form

$$\begin{aligned} \psi &= \begin{pmatrix} y_0 + L_0 & (y_n - L_n)\sigma^n \\ (y_n + L_n)\sigma^n & y_0 - L_0 \end{pmatrix}, \\ (I\psi) &= \begin{pmatrix} y_0 + L_0 & (y_n - L_n)\sigma^n \\ -(y_n + L_n)\sigma^n & -y_0 + L_0 \end{pmatrix}, \\ [c \partial/\partial t + J^n \partial/\partial x^n] &= \begin{pmatrix} c \partial/\partial t & \sigma^n \partial/\partial x^n \\ \sigma^n \partial/\partial x^n & c \partial/\partial t \end{pmatrix}, \\ [- (mc + A_0 e/c) + J^n A_n e/c] &= \begin{pmatrix} -(mc + A_0 e/c) & \sigma^n A_n e/c \\ \sigma^n A_n e/c & -(mc + A_0 e/c) \end{pmatrix}. \end{aligned}$$

According to the multiplication rules of octonionic matrices (18), or using the algebra of the basis units (2), equation (28) takes the form:

$$\begin{aligned} & \left[-mcy_0 - \frac{e}{c} A^\nu y_\nu + \hbar \partial^\nu L_\nu \right] + \left[-mcL_0 - \frac{e}{c} A^\nu L_\nu + \hbar \partial^\nu y_\nu \right] I + \\ & + \left[-mcy^i + \frac{e}{c} [Ay]^{i0} + \hbar F^{i0} + \epsilon^{ijk} \left(\hbar f_{jk} + \frac{e}{c} A_j L_k \right) \right] J_i + \\ & + \left[-mcL^i + \frac{e}{c} [AL]^{i0} + \hbar f^{i0} - \epsilon^{ijk} \left(\hbar F_{jk} + \frac{e}{c} A_j y_k \right) \right] j_i = 0, \end{aligned} \quad (29)$$

where $\nu = 0, 1, 2, 3$ and $i, j, k = 1, 2, 3$.

Now we have all the tools to reproduce the Dirac equations. Equating to zero coefficients in front of octonionic basis units in (29) we have a system of eight equations. Subtracting the pairs of these equations by the rules $(J_3 - j_3)$, $(j_2 - j_1)$, $(1 - J_2)$ and $(J_1 - I)$ we arrive at a system of four 'complex' equations

$$\begin{aligned}
& \frac{e}{c}[-A_0(y_0 + iL_3) + A_1(y_1 + iy_2) + A_2(y_2 - iy_1) + A_3(y_3 + iL_0)] - \\
& -\hbar[\partial_0(L_3 - iy_0) - \partial_1(y_2 - iy_1) + \partial_2(y_1 - iy_2) - \partial_3(L_0 - iy_3)] = cm(y_0 + iL_3), \\
& \frac{e}{c}[-A_0(-L_2 + iL_1) + A_1(y_3 + iL_0) - A_2(L_0 - iy_3) - A_3(y_1 + iy_2)] - \\
& -\hbar[\partial_0(L_1 + iL_2) - \partial_1(L_0 - iy_3) - \partial_2(y_3 + iL_0) + \partial_3(y_2 - iy_1)] = cm(iL_1 - L_2), \\
& \frac{e}{c}[-A_0(y_0 + iL_3) + A_1(y_1 + iy_2) + A_2(y_2 - iy_1) + A_3(y_3 + iL_0)] + \\
& +\hbar[\partial_0(L_0 - iy_3) - \partial_1(L_1 + iL_2) - \partial_2(L_2 + iL_1) - \partial_3(L_3 + iy_0)] = cm(y_3 + iL_0), \\
& \frac{e}{c}[-A_0(y_0 + iL_3) + A_1(y_1 + iy_2) + A_2(y_2 - iy_1) + A_3(y_3 + iL_0)] + \\
& +\hbar[\partial_0(y_2 - iy_1) - \partial_1(L_3 - iy_0) - \partial_2(y_0 + iL_3) + \partial_3(L_1 + iL_2)] = cm(y_1 + iy_2).
\end{aligned} \tag{30}$$

Introducing the complex bi-spinor (23) and gamma matrices (22) this system is easy to rewrite in matrix form, which exactly coincides with the standard Dirac equations (21).

To define the probability amplitudes we need a real modulus function corresponding to a positively defined probability. It cannot be the norm of the 'wave-function' (26), since the norm of split octonions is not positively defined and thus cannot give a satisfactory interpretation as a probability. However, the product

$$N = \frac{1}{2} [\psi\bar{\psi} + (\psi\bar{\psi})^*] = \frac{1}{2} [\psi\bar{\psi} + \psi^\dagger\psi^*] = y_0^2 + y_n y^n + L_n L^n + L_0^2, \tag{31}$$

can serve as the probability amplitude similar to standard Dirac theory. The product (31) is analogous to the Hermitian norm introduced in [23].

4 Concluding Remarks

In this paper it was shown that the Dirac equations can be written using the algebra of split octonions over the field of real numbers. As distinct from the similar models we do not use any complex numbers or bi-spinors.

We wish to stress that because of non-associativity our model is not equivalent to the standard Dirac theory. Without fixing the order of multiplications the products of wave-functions, corresponding to some physical process, will give not single-valued results. This property, when left and right products are not equivalent, in physical applications is natural to connect with the time asymmetry and causality.

Differences with standard theory arise also when we try to determine what kind of number should be used for the probability amplitudes. To formulate positively defined probability amplitudes four different split octonions (transforming into each other by discrete transformations) are necessary, rather than two complex wave functions corresponding to particles and antiparticles in usual Dirac theory. This means that in the octonionic model standard classification scheme of particles should be revised.

Note that the Hermitian norm (31), which we want to associate with the probability amplitude, exhibits more symmetry than the standard norm of split octonions (having $(4 + 4)$ signature). It is known that the automorphism group of ordinary octonions is the smallest exceptional Lie group - G_2 [5], while the automorphism group of split octonions is the real, non-compact form of G_2 . Some general results about the real, non-compact G_2 and its subgroup structure can be found in [24].

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